§1 Rational Functions

> *Curve sketching is a very important and misunderstood skill. It is much more mathematically sophisticated than plotting a curve; I explain to classes that plotting is a baby's game, like when you have to join the dots to get a picture of an elephant. Curve-sketching, by contrast, means being able to deduce the salient features of a curve, with a minimum of calculation.* – Owen Toller

We will see in this chapter that curve sketching can very much be a case of joining the dots, but in doing so we deduce some very salient features of a curve.

§1.1 The Method: ZISTA

When sketching $y = f(x)$ for any function, it is a good idea to consider *ZISTA*.

1. **Zeros** (Axes intercepts). When is $x = 0$ or $y = 0$
2. **Infinities** What is the behaviour as $x \to \pm\infty$ (or $y \to \pm\infty$)
3. **Sign** Where does $f(x)$ change sign?
4. **Turning points** Where are they?
5. **Asymptotes** Horizontal, Vertical and Oblique!

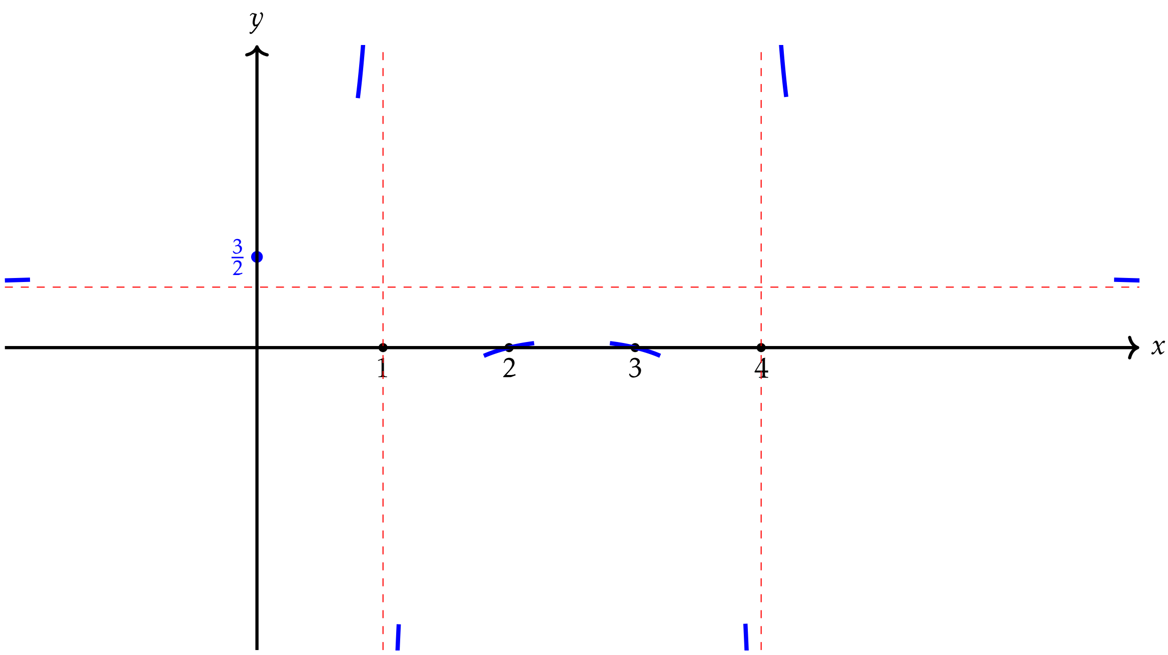
If our rational function is factorised we can often get many of these pieces of information simultaneously, for example:

$$f(x) = \frac{(x-a_1)(x-a_2)\cdots(x-a_k)}{(x-b_1)(x-b_2)\cdots(x-b_l)}$$

This function clearly as roots at $a_1, a_2, \cdots a_k$ and vertical asymptotoes at $b_1, b_2, \cdots, b_k$, it will change sign at all these points a_i and b_i (assuming we have no repeated roots).

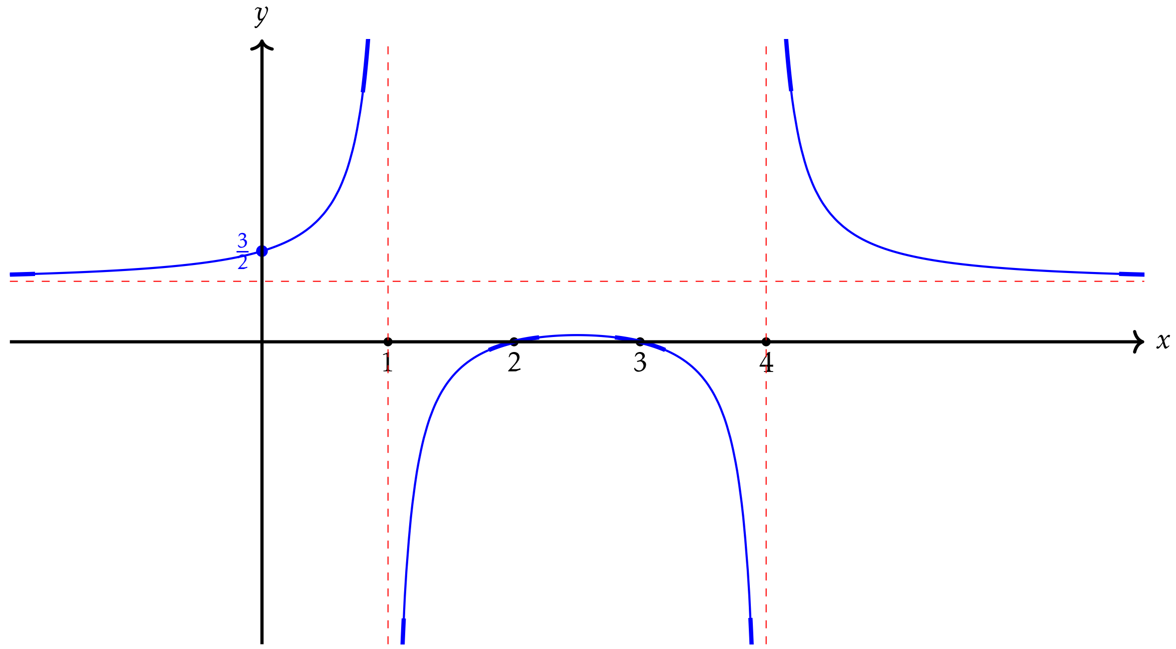
Example

Sketch $y = \dfrac{(x-2)(x-3)}{(x-1)(x-4)}$



1. **Zeros**: at $x = 2, x = 3$ (and when $x = 0$, $y = \frac{3}{2}$.
2. **Infinities**: this is a horizontal asymptote at $y = 1$
3. **Sign**: The sign changes at $1, 2, 3$ and 4
4. **Turning points**: The function is symmetric in $\frac{5}{2}$, so it will be at $(\frac{5}{2}, \frac{1}{9})$ (but we wont worry too much about this.
5. **Asymptotes**: We've already seen the horizontal one, the vertical ones will appear at $x = 1$ and $x = 4$

Finally, we join all our points together



A mathematician might ask 'how do we know there's no more turning points?'. There are several ways to see this. There are many nice arguments as to why this might be:

- The function is symmetric in $x = \frac{5}{2}$, and if there were more turning points we would have a horizontal line would intersect the graph > 2 times (which is not possible for a ratio of two quadratics - multiply it out and see you end up with a quadratic with more than 2 roots!)
- Suppose $\frac{(x-2)(x-3)}{(x-1)(x-4)} = \lambda \Leftrightarrow 0 = (\lambda-1)x^2 - 5(\lambda-1)x + 4\lambda - 6 \Leftrightarrow 25(\lambda-1)^2 - 4(\lambda-1)(4\lambda-6) \geq 0 \Leftrightarrow \lambda \leq \frac{1}{9}, \lambda > 1$. (Notice that when $\lambda = 1$ our argument about discriminants needs work!)
- Differentiate...

Example (CCE 1928 Paper 1 Q10)

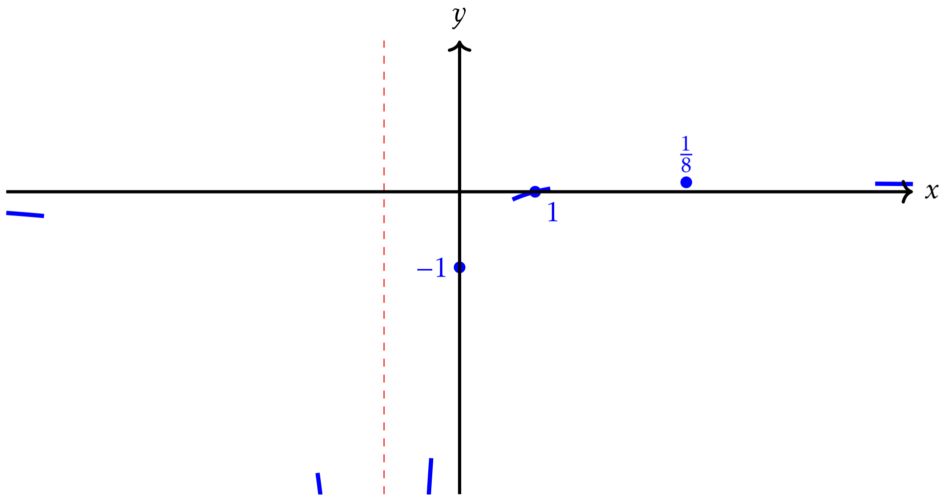
Sketch the graph

$$y = \frac{x-1}{(x+1)^2},$$

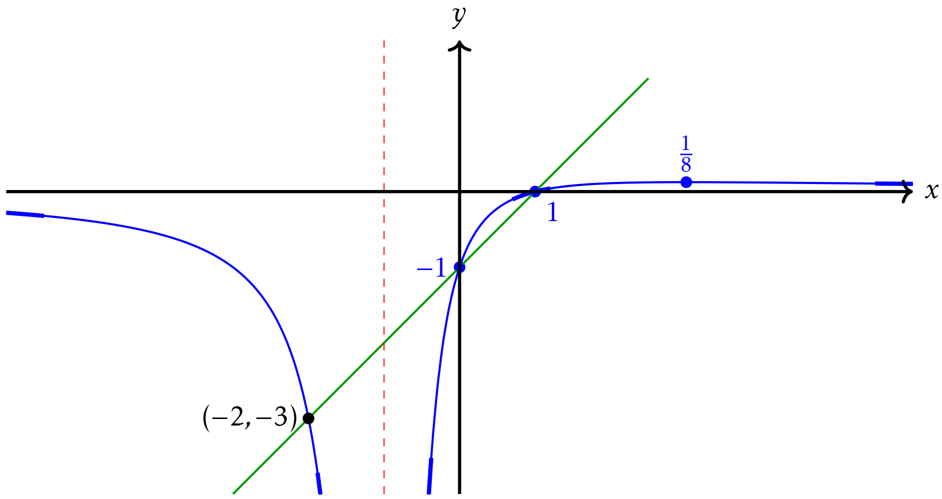
and find the point at which the line which joins the points where the curve meets the axes meets the curve again.

1. **Zeros**: When $y = 0$, $x = 1$. When $x = 0$, $y = -1$.
2. **Infinities**: As $x \to \pm\infty$, $y \to 0$. For large positive x, $y > 0$; for large negative x, $y < 0$.
3. **Sign**: The denominator $(x+1)^2$ is always positive. The function changes sign only at $x = 1$. It is negative for $x < 1$ (except at the asymptote) and positive for $x > 1$.
4. **Turning points**: $\frac{dy}{dx} = \frac{(x+1)^2 - 2(x+1)(x-1)}{(x+1)^4} = \frac{3-x}{(x+1)^3}$. There is a maximum at $(3, \frac{1}{8})$ and no other turning points.
5. **Asymptotes**: Vertical asymptote at $x = -1$. Horizontal asymptote at $y = 0$.

First, we plot the "dots" and the behaviour near key points:



Finally, we join the segments together and add the line. The line connecting $(1, 0)$ and $(0, -1)$ is $y = x - 1$.



To find the intersection, we solve:

$$x - 1 = \frac{x-1}{(x+1)^2} \implies (x-1)\left(1 - \frac{1}{(x+1)^2}\right) = 0$$

This gives $x = 1$ (intercept), or $(x+1)^2 = 1$. The latter yields $x+1 = 1 \implies x = 0$ (intercept) and $x+1 = -1 \implies x = -2$. At $x = -2$, $y = -2 - 1 = -3$. Thus, the line meets the curve again at $(-2, -3)$.

Example (CCE 1914 Paper 1 Q5)
Sketch the graph

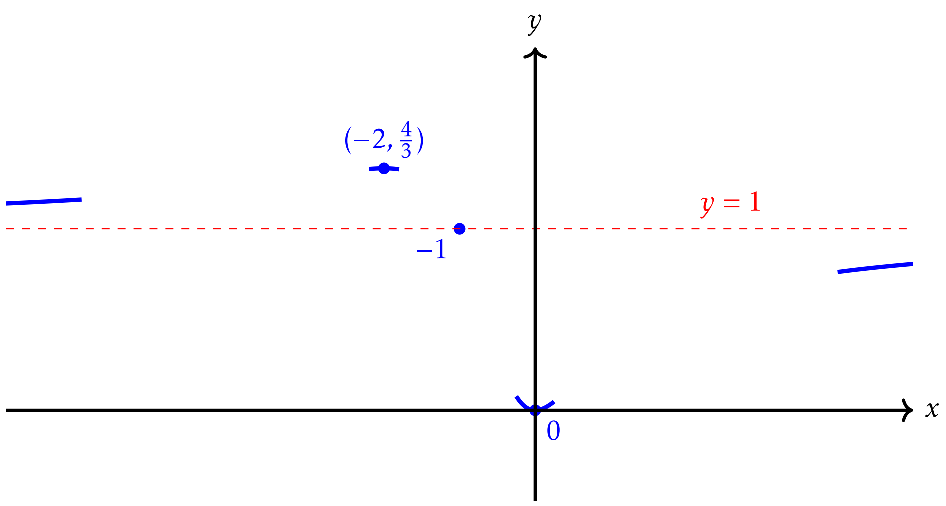
$$y = \frac{x^2}{x^2 + x + 1}$$

1. **Zeros**: At $x = 0$ only.
2. **Infinities**: As $x \to \pm\infty$, the x^2 terms dominate, so $y \to \frac{x^2}{x^2} = 1$. There is a horizontal asymptote at $y = 1$.
3. **Sign**: Since $x^2 \geq 0$ and the discriminant of the denominator is $1^2 - 4 \cdot 1 \cdot 1 = -3 < 0$ (meaning $x^2 + x + 1$ is always positive), the function is $y \geq 0$ for all x.
4. **Turning points**: Differentiating using the quotient rule:

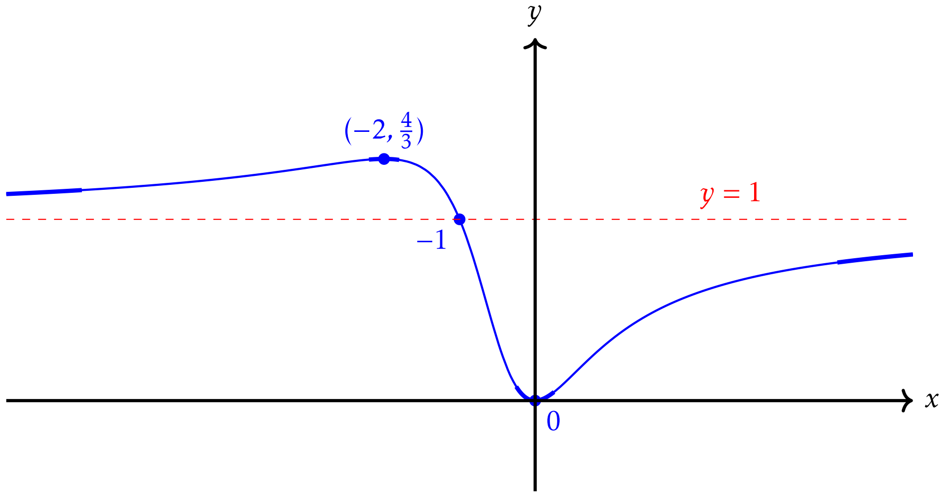
$$\frac{dy}{dx} = \frac{(x^2+x+1)(2x) - x^2(2x+1)}{(x^2+x+1)^2} = \frac{2x^3 + 2x^2 + 2x - 2x^3 - x^2}{(x^2+x+1)^2} = \frac{x^2 + 2x}{(x^2+x+1)^2}$$

   Setting $\frac{dy}{dx} = 0$ gives $x(x+2) = 0$.
   - At $x = 0$, $y = 0$ (a local minimum).
   - At $x = -2$, $y = \frac{4}{4-2+1} = \frac{4}{3}$ (a local maximum).
5. **Asymptotes**: There are no vertical asymptotes as the denominator has no real roots. There is a horizontal asymptote at $y = 1$. Note that the curve crosses the asymptote when $\frac{x^2}{x^2+x+1} = 1 \implies x^2 = x^2 + x + 1 \implies x = -1$.

First, we identify the key points and behaviour:



Finally, we join the dots to complete the sketch:


y
$(-2, \frac{4}{3})$
$y = 1$
-1
x
0

Example

Sketch the graph

$$y = \frac{x(x+1)^2}{(x-1)^2}$$

1. **Zeros**: $y = 0 \Rightarrow x(x+1)^2 = 0$. So $x = 0$ and $x = -1$.
   - At $x = -1$, the factor is squared, so the graph *touches* the x-axis (a repeated root).
   - At $x = 0$, it is a simple root, so the graph *crosses* the origin.
2. **Infinities**:
   - As $x \to 1$, the denominator $(x-1)^2 \to 0^+$. Since the numerator approaches $1 \cdot 2^2 = 4 > 0$, $y \to +\infty$ on both sides of $x = 1$.
   - As $x \to \pm\infty$, the degree of the numerator (3) is one higher than the denominator (2), indicating an oblique asymptote. Performing algebraic division:

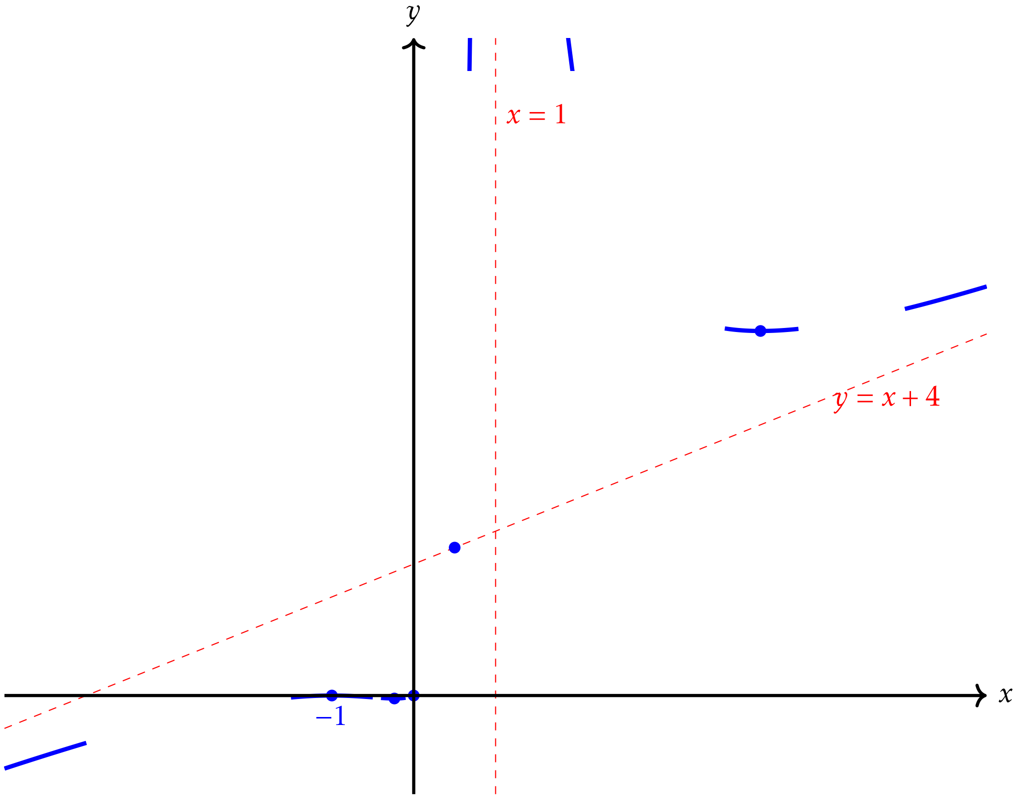
     $$y = \frac{x^3 + 2x^2 + x}{x^2 - 2x + 1} = x + 4 + \frac{8x - 4}{(x-1)^2}$$

     So, $y = x + 4$ is an oblique asymptote.
3. **Sign**: The term $(x+1)^2$ is always non-negative, and $(x-1)^2$ is always positive (for $x \neq 1$). The sign of y depends solely on x.
   - $y < 0$ for $x < 0$ (except at $x = -1$).
   - $y > 0$ for $x > 0$.
4. **Turning points**: Differentiating (using the quotient rule or logarithmic differentiation on $y = \frac{x(x+1)^2}{(x-1)^2}$) yields:

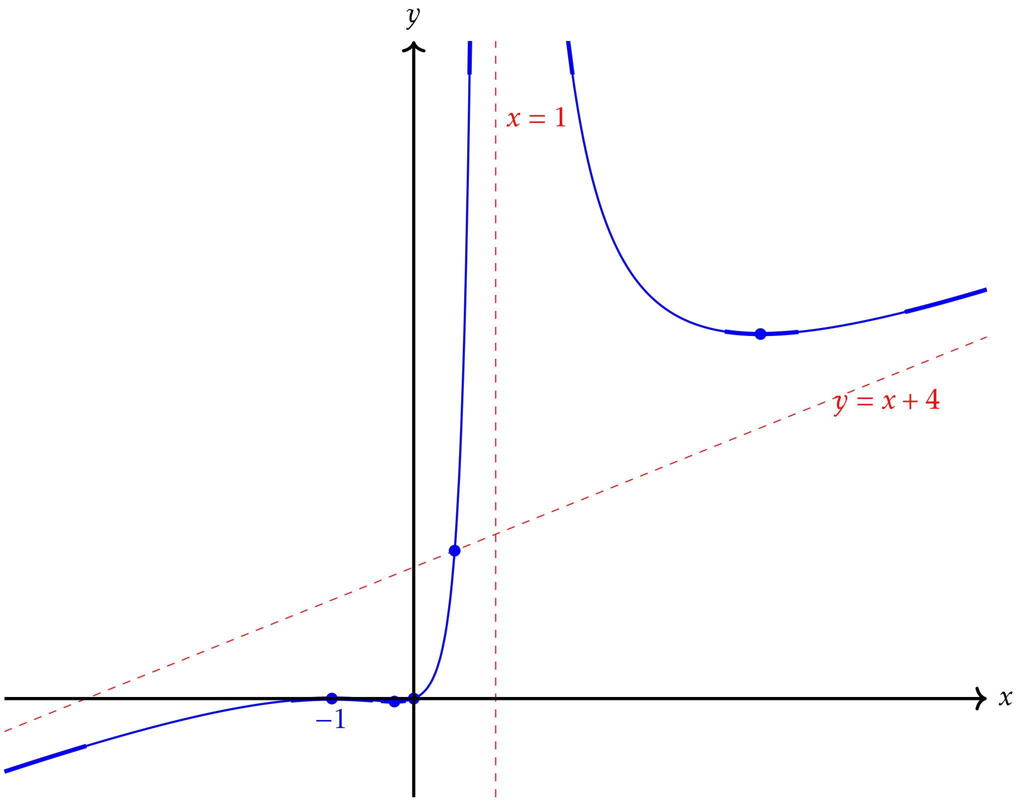
   $$\frac{dy}{dx} = \frac{(x+1)(x^2 - 4x - 1)}{(x-1)^3}$$

   $\frac{dy}{dx} = 0$ implies $x = -1$ or $x^2 - 4x - 1 = 0$.
   - $x = -1$ is a local maximum (we know $y = 0$ here and y is negative nearby).
   - Roots of $x^2 - 4x - 1$: $x = 2 \pm \sqrt{5}$.
5. **Asymptotes**: Vertical at $x = 1$. Oblique at $y = x + 4$.
6. **Crossing**: The curve crosses its oblique asymptote when the remainder term is zero: $8x - 4 = 0 \Rightarrow x = \frac{1}{2}$.

First, we set up the scaffolding: the asymptotes, the behaviour at the limits, and the known points.



Now we join the dots



Example (CCE 1925 Paper 1 Q10)
Sketch
$$y = 10\frac{x^2+3x}{2x^2+13x-7}.$$

First, we factorise the expression to make the roots and asymptotes clear:

$$y = \frac{10x(x+3)}{(2x-1)(x+7)}$$

1. **Zeros**: The numerator is zero when $x = 0$ or $x = -3$. Thus, the curve passes through $(0,0)$ and $(-3,0)$.
2. **Infinities**: As $x \to \pm\infty$, the highest powers dominate: $y \approx \frac{10x^2}{2x^2} = 5$. There is a horizontal asymptote at $y = 5$.

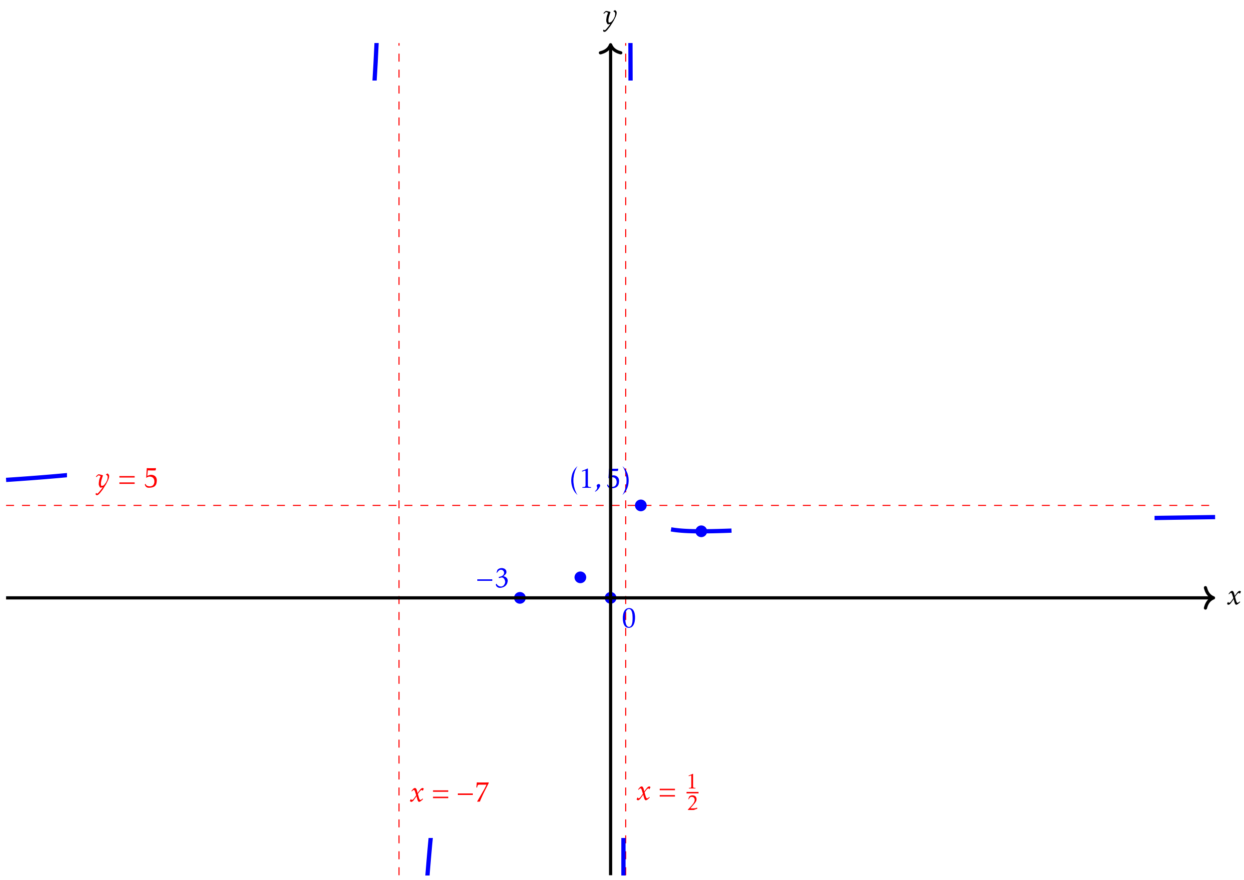
   Does it cross the asymptote? Set $y = 5$:

   $$5 = \frac{10x^2+30x}{2x^2+13x-7} \implies 10x^2+65x-35 = 10x^2+30x \implies 35x = 35 \implies x = 1.$$

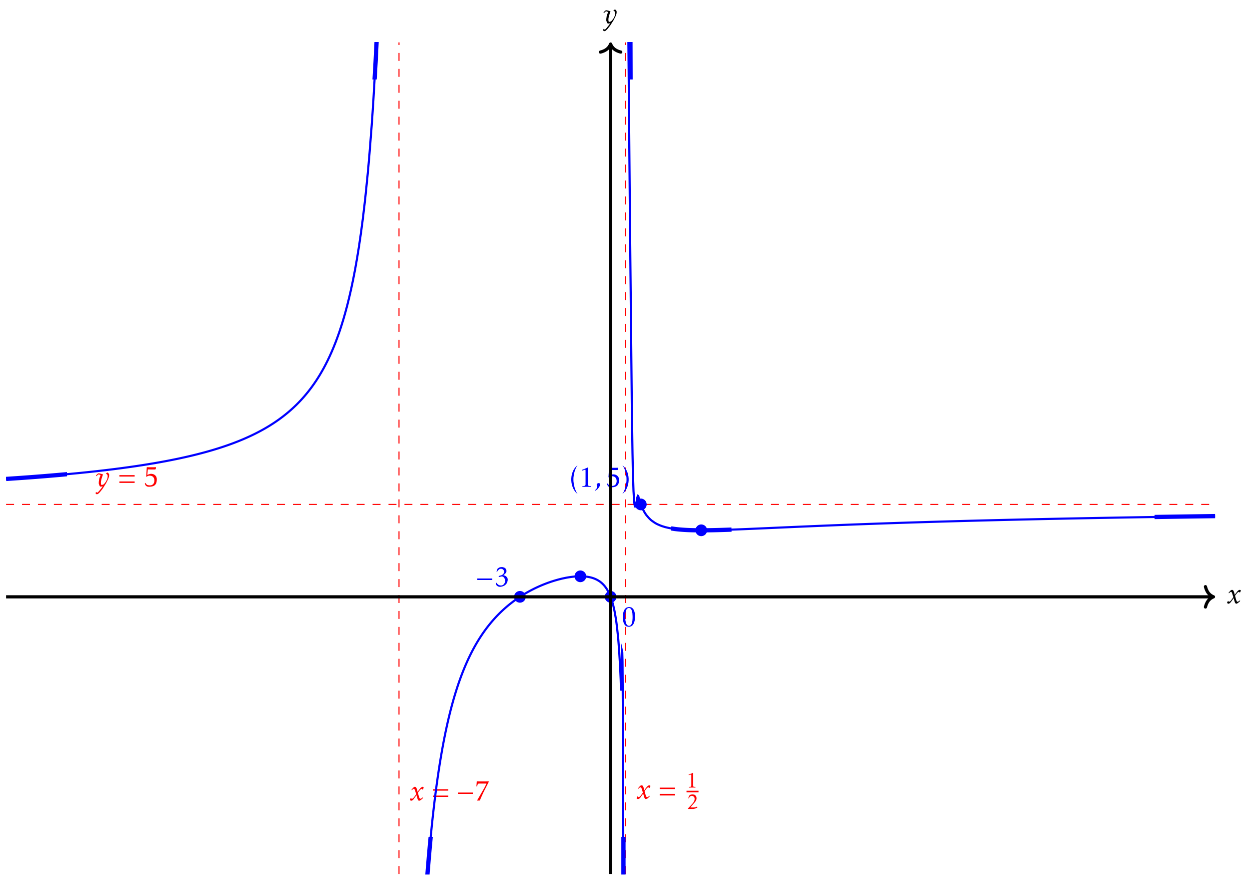
   So the curve crosses the horizontal asymptote at $(1,5)$.
3. **Sign**: The graph changes sign at $-7, -3, 0, \frac{1}{2}$.
4. **Turning points**:

   $$\begin{aligned} & y = \frac{10x^2+30x}{2x^2+13x-7} \\ \Leftrightarrow \quad & 0 = (10-2y)x^2 + (30-13y)x + 7y \\ \text{has a sol'n} \Leftrightarrow \quad & 0 \le \Delta = (30-13y)^2 - 4\cdot(10-2y)\cdot 7y \\ & \quad = 225y^2 - 1060y + 900 \\ & \quad = 5(9y-10)(5y-18) \end{aligned}$$

   Therefore $y \ge \frac{18}{5}$ or$y \le \frac{10}{9}$
5. **Asymptotes**: Vertical asymptotes at $x = -7$ and $x = \frac{1}{2}$. Horizontal at $y = 5$.



And again joining the dots



Example (CCE 1937 Paper 1 Q10)

Sketch the curve

$$y(y+1)(y+2)-(x-2)x(x+2)=0$$

and prove that the point $(0,-1)$ is a point of inflection.

We can very quickly discover many things about this graph:

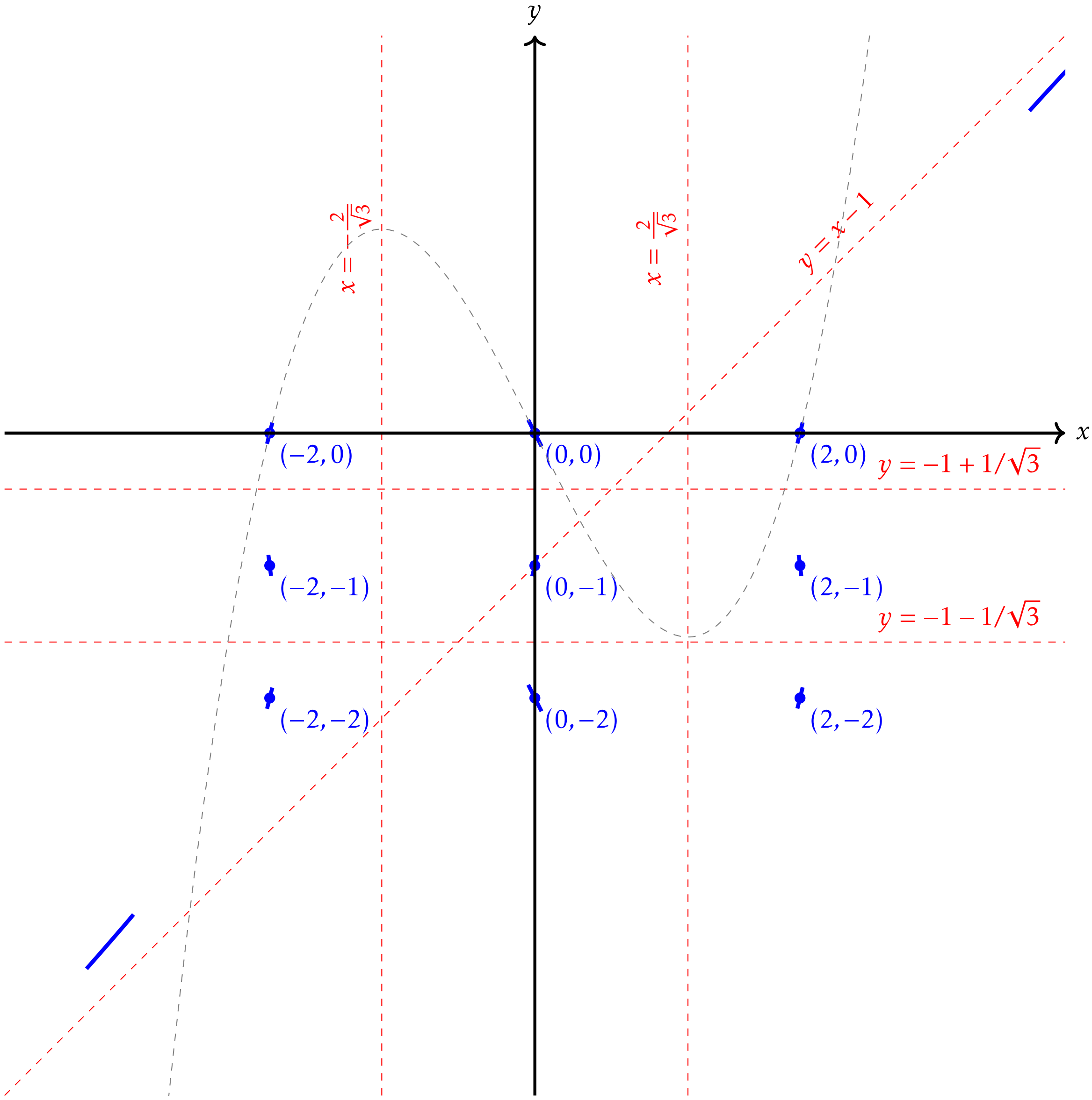
(a) When $y=0,-1,-2$, $x=2,0,-2$, so we have 9 points on our curve immediately.

(b) When $x\to\infty$, $y\approx x$ (more precisely there is an asymptote $y=x-1$ which we can only cross once at $(0,-1)$).

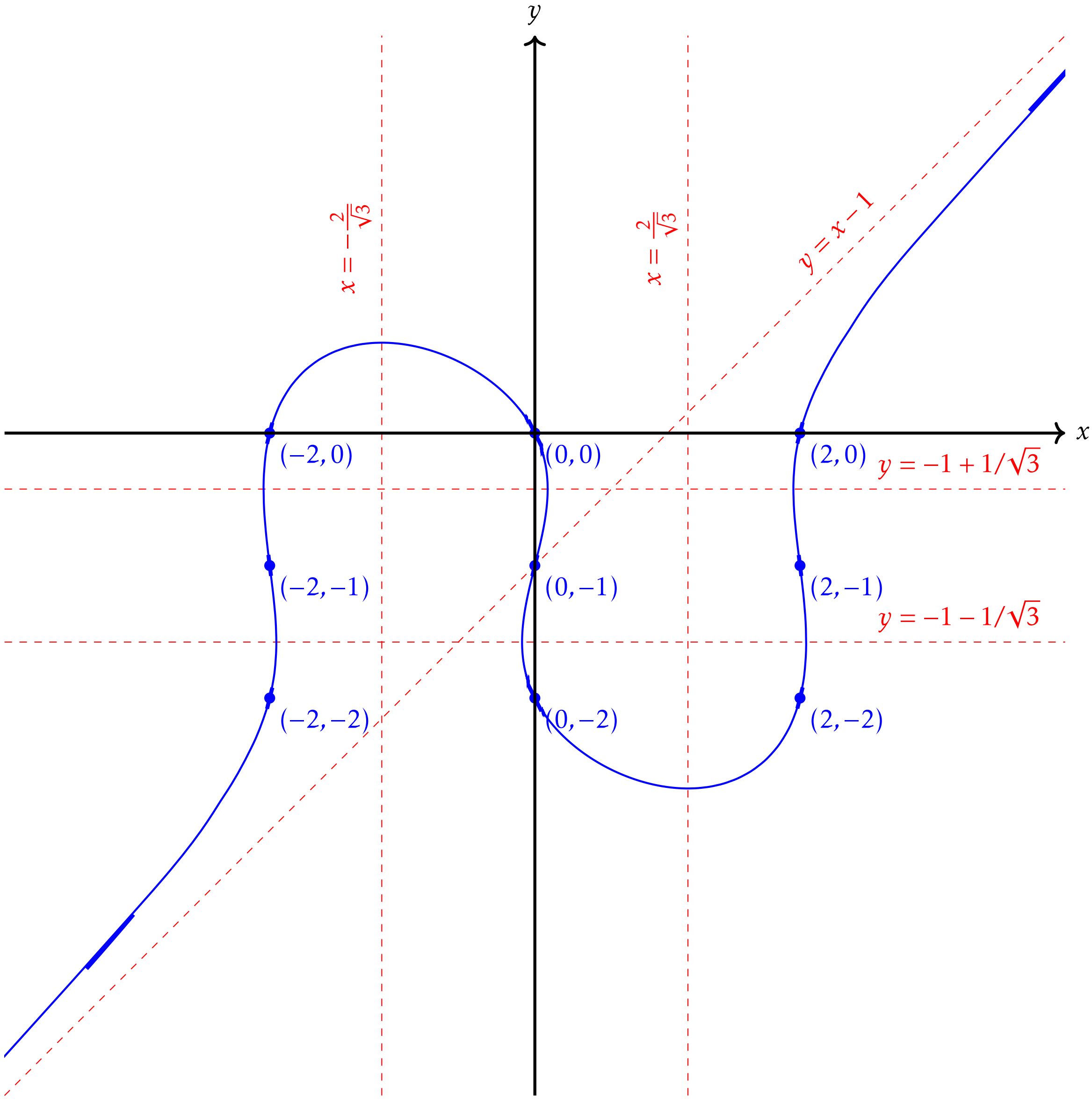
(c) $(3y^2+6y+2)y'=3x^2-4$ so we have horizontal tangents at $x=\pm\frac{2}{\sqrt{3}}$ and vertical tangents at $y=-1\pm\frac{1}{\sqrt{3}}$

(d)
- Around $y=0$ we have $(x-2)x(x+2)\approx 2y$
- Around $y=-1$ we have $(x-2)x(x+2)\approx -(y+1)$
- Around $y=-2$ we have $(x-2)x(x+2)\approx 2(y+2)$

So we can see the 'direction' our curve must be heading in around those points



And then we join the dots.



y
x
$x = -\frac{2}{\sqrt{3}}$
$x = \frac{2}{\sqrt{3}}$
$y = x - 1$
$(-2, 0)$
$(0, 0)$
$(2, 0)$
$y = -1 + 1/\sqrt{3}$
$(-2, -1)$
$(0, -1)$
$(2, -1)$
$y = -1 - 1/\sqrt{3}$
$(-2, -2)$
$(0, -2)$
$(2, -2)$

§1.1.1 Exercises

1. Sketch the curve $y = \frac{(x-2)(x-3)}{(x-1)(x-4)}$.
   a) State the equations of the vertical asymptotes.
   b) Find the coordinates of the points where the curve crosses the axes.
   c) By performing polynomial division, find the equation of the horizontal asymptote.
   d) Determine the equation of the tangent at the point where the curve crosses the y-axis.

2. a) Sketch the curve $y = \frac{x}{(x+1)(x+2)}$ and determine the maximum and minimum values of its ordinate.
   b) Hence or otherwise, sketch $y^2 = \frac{x}{(x+1)(x+2)}$.

   CCE 1950 Paper 3 Q8

3. Sketch the graph of the function given by

$$f(x) = \frac{x-a}{x(x-2)}$$

   where a is a constant, in each of the following cases:
   a) $0 < a < 2$
   b) $a > 2$

   Explain briefly how you deduce the salient features of these graphs. *CCE 1975 Paper 1 Q8*

4. Sketch the curve with cartesian equation

$$y = \frac{2x(x^2-5)}{x^2-4}$$

   and give the equations of the asymptotes and of the tangent to the curve at the origin. Hence determine the number of real roots of the following equations:
   (i) $3x(x^2-5) = (x^2-4)(x+3)$;
   (ii) $4x(x^2-5) = (x^2-4)(5x-2)$;
   (iii) $4x^2(x^2-5)^2 = (x^2-4)^2(x^2+1)$.

   STEP 2006 Paper 3 Q1

5. The equation of a curve is $y = f(x)$ where

$$f(x) = x - 4 - \frac{16(2x+1)^2}{x^2(x-4)}.$$

   (i) Write down the equations of the vertical and oblique asymptotes to the curve and show that the oblique asymptote is a tangent to the curve.
   (ii) Show that the equation $f(x) = 0$ has a double root.
   (iii) Sketch the curve.

   STEP 2004 Paper 3 Q2

6. a) Find the asymptotes of the curve $y^2(x^2-1) = x^3$.
   b) Sketch the curve. (Hint: Consider symmetry and the domain where y is real).

   CCE 1957 Paper 3 Q8

7. Discuss the general existence of maxima or minima of the function

$$y = \frac{x^2 + 2ax + b}{x^2 + 2Ax + B}$$

and sketch the various possible forms which the graph of the function may have for different values of the constant coefficients. Illustrate your remarks by reference to the functions:

$$\text{(i) } \frac{x^2}{x^2 + x + 1}, \quad \text{(ii) } \frac{x^2 - 1}{x^2 - 4}, \quad \text{(iii) } \frac{x(x-2)}{(x-1)(x-3)}.$$

CCE 1916 Paper 3 Q10

8. Sketch the curve

$$y = \frac{x^2}{x^2 + 3x + 2}$$

By means of the line $y + 8 = m(x + 1)$, or otherwise, find the number of real roots of the equation

$$m(x+1)^2(x+2) = (3x+4)^2$$

when m is a real constant which is (i) positive, (ii) negative. *CCE 1944 Paper 4 Q8*

9. The function f is defined by

$$\mathrm{f}(x) = \frac{(x-a)(x-b)}{(x-c)(x-d)} \qquad (x \neq c,\ x \neq d),$$

where a, b, c and d are real and distinct, and $a + b \neq c + d$. Show that

$$\frac{x\mathrm{f}'(x)}{\mathrm{f}(x)} = \left(1 - \frac{a}{x}\right)^{-1} + \left(1 - \frac{b}{x}\right)^{-1} - \left(1 - \frac{c}{x}\right)^{-1} - \left(1 - \frac{d}{x}\right)^{-1},$$

$(x \neq 0, x \neq a, x \neq b)$ and deduce that when $|x|$ is much larger than each of $|a|, |b|, |c|$ and $|d|$, the gradient of $\mathrm{f}(x)$ has the same sign as $(a + b - c - d)$. It is given that there is a real value of real value of x for which $\mathrm{f}(x)$ takes the real value z if and only if

$$[(c-d)^2 z + (a-c)(b-d) + (a-d)(b-c)]^2 \geqslant 4(a-c)(b-d)(a-d)(b-c).$$

Describe briefly a method by which this result could be proved, but do not attempt to prove it. Given that $a < b$ and $a < c < d$, make sketches of the graph of f in the four distinct cases which arise, indicating the cases for which the range of f is not the whole of $\mathbb{R}$. *STEP 1989 Paper 2 Q4*